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ON A RANDOM INTERVAL GRAPH AND THE MAXIMUM THROUGHPUT RATE IN THE SYSTEM $GI/G/1/0$

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Abstract

The paper gives an explicit expression for the expectation of the maximum attainable fraction of served customers in the long run for the single-server loss system $GI/G/1/0$, under the assumption of perfect information regarding the sequences $\{X_i, i=1,2,\cdots\}$ and $\{Y_i, i=1,2,\cdots\}$ of interarrival times and service times, respectively. A heavy traffic result for this fraction is obtained for the system $GI/M/1/0$. The general result is based on an analysis of the random interval graph corresponding to the random intervals $\{[T_i, T_i+Y_i), i=1,2,\cdots\}$, in which $\{T_i\}$ denotes the sequence of arrival epochs.

STABLE SET; SINGLE-SERVER LOSS SYSTEM; OPTIMISATION

Introduction

Consider a single-server loss system with renewal input $\{X_i, i=1,2,\cdots\}$ and generally distributed service times $\{Y_i, i=1,2,\cdots\}$, where Y_i is the service time of the ith customer and $T_i=X_1+X_2+\cdots+X_{i-1}$ is its arrival epoch ($T_1=0$). Both families of random variables are assumed to be independent, and EX_i and EY_i are finite. The intervals $\{[T_i, T_i+Y_i)\}$ represent the service intervals required by successive customers.

Suppose one wants to control the admission to the system in order to maximize the throughput of the system in the long run. Every time a customer arrives when the server is idle one can decide whether or not to accept this customer, given the state information. Using an on-line control the decision depends on the available information brought in by the arriving customer. If his service time is unknown to the controller the best one can do is to accept him, leading to the classical $GI/G/1/0$ loss system. On the other hand, if the service times are known to the controller the optimal input control is a critical number policy. In fact, this optimisation model is a special case of the well-known streetwalker's dilemma introduced by Lippman and Ross, see Ross [6], which can be analysed using semi-Markov decision theory. In Nawijn [5] we consider the case in which the controller has partial information over the future, in particular, it is assumed that at each arrival epoch $T_i(\omega)$, the service time $Y_i(\omega)$ of the present customer and the arrival epoch $T_{i+1}(\omega)$ and the service time $Y_{i+1}(\omega)$ of the next customer are known. Here we consider the extreme case

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that one has perfect information about the future, that is one exactly knows the set $\{[T_i(\omega), T_i(\omega)+Y_i(\omega)), i=1, 2, \cdots\}$ of service intervals.

Let $Z_n(\omega)$ be the maximum number of customers that can be served among the first n arriving customers under perfect information. We will investigate the limiting behaviour of the random variable Z_n/n as $n\rightarrow\infty$. Its expectation gives the maximal attainable fraction of served customers for the $GI/G/1/0$ system and acts as an upper bound for any system with partial information, for which no explicit results exist. The analysis of the random variable Z_n exploits an equivalence relation between the $GI/G/1/0$ system and a random interval graph. Based on the family of random service intervals a random interval graph G_∞ is defined with vertex set $\mathbb{N}=\{i:i=1, 2, \cdots\}$ corresponding to the intervals, such that (i, j) is an edge in G_∞ if the two corresponding intervals intersect. By G_n we denote the random interval graph on the vertex set $\{1, 2, \cdots, n\}$ corresponding to the first n arriving customers. A stable set of a graph is a set of mutually non-adjacent vertices, i.e. there is no edge connecting any two vertices in this set. Clearly, a stable set corresponds to a set of schedulable customers without service conflicts. Hence, the random variable Z_n is equivalent to the cardinality of a maximum stable set of G_n.

In the next section we give an analysis of the random graph G_n and determine, among other things, the limiting distribution of Z_n/n. In the last section we consider the system $GI/M/1/0$ in heavy traffic.

1. The random interval graph

We first introduce some notions and results from the theory of interval graphs, which can be found in e.g. Golumbic [4]. Let $H=(V, E)$ be a (deterministic) interval graph with vertex set V, $|V|=n$, and edge set E. Let $\mathrm{Adj}\,(v)=\{u\in V \mid (u, v)\in E\}$ denote the set of neighbours of a vertex v. An ordering $\sigma=(j_1, j_2, \cdots, j_n)$ of the vertices of V is called *a perfect elimination scheme* for H if for each $j_i\in V$ the vertices in the set $\Gamma_{j_i}=\{j_k\in \mathrm{Adj}\,(j_i) \mid k>i\}$ are mutually adjacent, that is the induced subgraph on Γ_{j_i} is a *clique*. Observe that a clique corresponds to a set of service intervals having a common non-empty intersection. The algorithm to find a maximum stable set $\{v_1, v_2, \cdots, v_t\}\subseteq V$, given a perfect elimination scheme σ, is specified recursively as follows: $v_1=\sigma(1)$; $v_i=\sigma(k_{i-1})$ where $k_{i-1}=\min\{j:\sigma(j)\notin\bigcup_{r=1}^{i-1}\Gamma_{v_r}\}$, $2\leqq i\leqq t-1$, and v_t is such that all vertices in σ following v_t are elements of $\bigcup_{r=1}^{t}\Gamma_{v_r}$. The fact that the algorithm simultaneously generates a stable set of cardinality t and t cliques $\{v_1\}\cup\Gamma_{v_1}$, $\{v_2\}\cup\Gamma_{v_2}, \cdots, \{v_t\}\cup\Gamma_{v_t}$ covering all the vertices of H proves that the stable set has maximum cardinality (and that the clique covering uses the minimum number of cliques).

Now let a realisation of the service intervals $\{[T_i(\omega), T_i(\omega)+Y_i(\omega)), r=1, 2, \cdots\}$ be known and let $G_n(\omega)$ be the interval graph representation of the first n intervals. To determine the maximum stable set of $G_n(\omega)$ one needs an elimination scheme. Although several such schemes exist the best one serving the analysis is

$\sigma = (j_1, j_2, \cdots, j_n)$ generated by ordering the corresponding intervals according to increasing values of $T_i(\omega) + Y_i(\omega)$. It is readily verified that this is indeed a perfect scheme. To analyse Z_n let us introduce the random variables

$$v(1) = \operatorname{argmin}\{T_i + Y_i,\ i = 1, 2, \cdots, n\}, \qquad D_n = |\Gamma_{v(1)}| + 1.$$

Observe that D_n equals the number of intervals intersecting the interval $[T_{v(1)}, T_{v(1)} + Y_{v(1)})$ plus 1, that is the cardinality of the clique $\Gamma_{v(1)} \cup \{v(1)\}$. In terms of our loss system $D_n - 1$ counts the number of customers having service conflicts with customer $v(1)$ having the earliest possible service completion time.

Let

$$\Pr\{X_i \leqq t\} = A(t), \qquad \Pr\{Y_i \leqq t\} = B(t), \qquad t \geqq 0,$$

with $A(0+) = 0$ and $B(0+) = 0$.

Theorem 1. *Let* $S_i = \sum_{j=1}^{i} X_j$, $i \geqq 1$, *then*

$$(1)\qquad \begin{aligned} \Pr\{D_n = j\} &= E\left\{B(S_j)\prod_{i=1}^{j-1}[1 - B(S_i)]\right\}, \qquad 1 \leqq j \leqq n-1, \\ \Pr\{D_n = n\} &= E\left\{\prod_{i=1}^{n-1}[1 - B(S_i)]\right\}, \end{aligned}$$

the empty product being 1 *by definition.*

Proof. We start by considering the simultaneous distribution of $v(1)$ and D_n. First observe that $\Pr\{v(1) = k, D_n = j\} = 0$ if $j < k$, since $\{v(1) = k\}$ implies that the first $k - 1$ intervals intersect the kth interval, thus, we necessarily have $j \geqq k$. Let us assme that $1 \leqq k \leqq j \leqq n - 1$. Observe that

$$(2)\qquad \begin{aligned} \Pr\{v(1) = k, D_n = j\} &= \Pr\left[\bigcap_{i=1}^{k-1}\{T_i + Y_i > T_k + Y_k\} \cap \bigcap_{i=k+1}^{j}\{\{T_i < T_k + Y_k\}\right. \\ &\qquad \left. \cap \{T_i + Y_i > T_k + Y_k\}\} \cap \{T_{j+1} \geqq T_k + Y_k\}\right] \\ &= \Pr\left[\bigcap_{i=1}^{k-1}\{Y_i > Y_k + S_{i,k-1}\} \cap \bigcap_{i=k+1}^{j}\{Y_i > Y_k - S_{k,i-1}\}\right. \\ &\qquad \left. \cap \{S_{k,j-1} < Y_k\} \cap \{S_{k,j} \geqslant Y_k\}\right] \end{aligned}$$

where for notational convenience we write $S_{i,j} = \sum_{r=i}^{j} X_r$. By the independence assumptions we have that

$$(3)\qquad \begin{aligned} &\Pr\{v(1) = k, D_n = j \mid X_1 = x_1, \cdots, X_j = x_j;\ Y_k = y\} \\ &= \prod_{i=1}^{k-1}[1 - B(y + s_{i,k-1})] \prod_{i=k+1}^{j}[1 - B(y - s_{k,i-1})] \end{aligned}$$

for $s_{k,j-1} < y \leqq s_{k,j}$, with $s_{i,j} = x_i + \cdots + x_j$. Hence,

$$\begin{aligned} \Pr\{D_n = j\} &= \sum_{k=1}^{j} \int_{x_i=0}^{\infty} \cdots \int_{x_j=0}^{\infty} \int_{y=s_{k,j-1}}^{s_{k,j}} \prod_{i=1}^{k-1} [1 - B(y + s_{i,k-1})] \\ &\quad \times \prod_{i=k+1}^{j} [1 - B(y - s_{k,i-1})]\, dB(y)\, dA(x_1) \cdots dA(x_j) \\ &\stackrel{\mathrm{d}}{=} \sum_{k=1}^{j} \int_{x_1=0}^{\infty} \cdots \int_{x_j=0}^{\infty} J_k(x_1, \cdots, x_j)\, dA(x_1) \cdots dA(x_j). \end{aligned} \tag{4}$$

Now consider J_j, which is given by

$$J_j = \int_0^{x_j} \prod_{i=1}^{j-1} [1 - B(y + s_{i,j-1})]\, dB(y), \tag{5}$$

noting that empty products equal 1 and empty sums equal 0. Using integration by parts, one obtains

$$J_j = B(x_j) \prod_{i=1}^{j-1} [1 - B(s_{i,j})] - \int_0^{x_j} B(y)\, d_y \prod_{i=1}^{j-1} [1 - B(y + s_{i,j-1})].$$

Since

$$dy \prod_{i=1}^{j-1} [1 - B(y + s_{i,j-1})] = -\sum_{i=1}^{j-1} \prod_{\substack{r=1 \\ r \neq i}}^{j=1} [1 - B(y + s_{r,j-1})]\, d_y B(y + s_{i,j-1}) \tag{6}$$

it follows that

$$J_j = B(x_j) \prod_{i=1}^{j-1} [1 - B(s_{i,j})] + \sum_{i=1}^{j-1} \int_0^{x_j} \prod_{\substack{r=1 \\ r \neq i}}^{j-1} [1 - B(y + s_{r,j-1})] B(y)\, d_y B(y + s_{i,j-1}).$$

Introducing the new integration variable $u = y + s_{i,j-1}$, we obtain

$$\begin{aligned} J_j &= B(x_j) \prod_{i=1}^{j-1} [1 - B(s_{i,j})] \\ &\quad + \sum_{i=1}^{j-1} \int_{s_{i,j-1}}^{s_{i,j}} \prod_{r=1}^{i-1} [1 - B(u + s_{r,i-1})] \prod_{r=i+1}^{j-1} [1 - B(u - s_{i,r-1})] B(u - s_{i,j-1})\, dB(u). \end{aligned} \tag{7}$$

Substituting (7) into (4) shows that

$$\begin{aligned} \sum_{k=1}^{j} J_k(x_1, x_2, \cdots, x_j) &= B(x_j) \prod_{i=1}^{j-1} [1 - B(s_{i,j})] \\ &\quad + \sum_{k=1}^{j-1} \int_{s_{k,j-1}}^{s_{k,j}} \prod_{r=1}^{k-1} [1 - B(y + s_{r,k-1})] \prod_{r=k+1}^{j-1} [1 - B(y - s_{k,r-1})]\, dB(y). \end{aligned}$$

Again consider the last term $k = j - 1$ of the sum on the right, now given by

$$\int_{s_{j-1,j-1}}^{s_{j-1,j}} \prod_{r=1}^{j-2} [1 - B(y + s_{r,j-2})]\, dB(y).$$

This integral is similar to J_j in (5). Therefore, the above procedure can be repeated. It is straightforward to show in general, using integration by parts and relation (6), that

$$\sum_{k=1}^{j} J_k(x_1, \cdots, x_j) = \sum_{k=j-m}^{j} \left[B(s_{k,j}) \prod_{i=1}^{k-1} [1 - B(s_{i,j})] - B(s_{k,j-1}) \prod_{i=1}^{k-1} [1 - B(s_{i,j-1})] \right]$$
$$+ \sum_{k=1}^{j-m-1} \int_{s_{k,j-1}}^{s_{k,j}} \prod_{r=1}^{k-1} [1 - B(y + s_{r,k-1})] \prod_{r=k+1}^{j-m-1} [1 - B(y - s_{k,r-1})] \, dB(y)$$

which finally leads to

$$\sum_{k=1}^{j} J_k = \prod_{i=1}^{j} \left[B(s_{k,j}) \prod_{i=1}^{k-1} [1 - B(s_{i,j})] - B(s_{k,j-1}) \prod_{i=1}^{k-1} [1 - B(s_{i,j-1})] \right].$$

In order to express $\Pr \{D_n = j\}$ in its simplest form we rewrite this expression by collecting all terms containing the factor $B(s_{1,j})$:

$$\sum_{k=1}^{j} J_k = B(s_{1,j}) \left\{ 1 - \sum_{r=1}^{j-1} B(s_{r+1,j}) \prod_{i=1}^{r-1} [1 - B(s_{i+1,j})] \right\}$$
$$+ \sum_{r=0}^{j-2} B(s_{r+2,j}) \prod_{i=1}^{r} [1 - B(s_{i+1,j})] - \sum_{r=0}^{j-2} B(s_{r+1,j-1}) \prod_{i=1}^{r} [1 - B(s_{i,j-1})].$$

Observe that the last two sums on the right can be expressed as $f(x_2, x_3, \cdots, x_j)$ and $f(x_1, x_2, \cdots, x_{j-1})$, respectively. So, since the random variables $\{X_i\}$ are identically distributed, we obtain after taking the expectation with respect to the underlying probability distribution, see (4),

$$\Pr \{D_n = j\} = E\left[B(S_{1,j}) \left\{ 1 - \sum_{r=1}^{j-1} B(S_{j-r+1,j}) \prod_{i=r+1}^{j-1} [1 - B(S_{j-i+1,j})] \right\} \right]. \tag{8}$$

The partial sum $S_{i,j} = \sum_{r=i}^{j} X_r$ may, however, be replaced by a partial sum $S_{j-i+1} = X_1 + \cdots + X_{j-i+1}$, since the random variables X_i are exchangeable. It is easily proved by mathematical induction that

$$\prod_{i=1}^{n} [1 - a_i] = 1 - \sum_{i=1}^{n} a_i \prod_{j=i+1}^{n} [1 - a_j].$$

Application of this relation to (8) proves our assertion, where $\Pr \{D_n = n\}$ is obtained from the normalising condition.

It should be noticed that the probabilities $\Pr \{D_n = j\}$, $1 \leqq j \leqq n - 1$, are independent of n. Moreover, observe that the event $\{D_n = j\}$, $1 \leqq j \leqq n - 1$, is independent of events defined in terms of the random variables $\{X_i,\ i = j + 1, \cdots, n - 1\}$ and $\{Y_i,\ i = j + 1, \cdots, n\}$, as can be inferred from relation (2). Hence, $D_n + 1$ acts as a stopping time, which suggests to analyse the cardinality of a maximum stable set Z_n of the random graph G_n through the use of the theory of discrete-time renewal theory.

Let $\{U_i,\ i=1,2,\cdots\}$ be a sequence of discrete-valued i.i.d. random variables having probability distribution $\{p_j,\ j=1,2,\cdots\}$, defined by

$$p_j = E\left\{B(S_j)\prod_{i=1}^{j-1}[1-B(S_i)]\right\}, \qquad j=1,2,\cdots. \tag{9}$$

Apparently, we have $\Pr\{D_n=j\}=\Pr\{U_1=j\}$ as $n\to\infty$. Recalling the algorithm to determine a maximum stable set, we have the following interpretation of Z_n in terms of the renewal sequence $\{U_i\}$:

$$Z_n = 1+\max\left\{i: \sum_{k=0}^{i} U_k \leqq n,\ i\geqq 0\right\}, \qquad U_0=0,$$

that is, Z_n equals the number of renewals among the time epochs $1,2,\cdots,n$ including the initial one at time 1. Moreover, if $\{v(i),\ i=1,2,\cdots\}$ denotes the stable set generated by the perfect elimination scheme for G_∞, then $\{U_i,\ i=1,2,\cdots\}$ corresponds to $\{|\{v(i)\}\cup\Gamma_{v(i)}|,\ i=1,2,\cdots\}$, the cardinalities of the cliques generated by the elimination scheme.

In terms of the service system the customers $\{v(i)\}$ to be admitted to the system are generated recursively as follows: $v(i)$ is the customer with earliest possible service completion among the customers having no service conflicts with customer $v(i-1)$; U_i counts customer $v(i)$ plus the number of customers having service conflict with $v(i)$, that arrive after the service completion of customer $v(i-1)$.

From renewal theory we have the following main result, see for example Feller [1], p. 297.

Theorem 2. *If* $\sum j^2 p_j<\infty$, *then*

$$\lim_{n\to\infty}\frac{1}{n}EZ_n=\frac{1}{1+\theta}, \qquad \theta \stackrel{\mathrm{d}}{=} \sum_1^\infty jp_j=\sum_{j=1}^\infty E[1-B(S_j)]$$

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{Var}(Z_n)=\frac{v}{(1+\theta)^3}, \qquad v\stackrel{\mathrm{d}}{=}\operatorname{Var}U_i$$

$$\lim_{n\to\infty}\Pr\left\{\frac{Z_n-n/(1+\theta)}{(vn)^{1/2}(1+\theta)^{-3/2}}<x\right\}=\Phi(x),$$

where $\Phi(\ )$ *denotes the standard normal distribution.*

It is interesting to observe that, apart from the interpretation of p_j as the probability that a clique $\Gamma_{v(i)}\cup\{v(i)\}$ in G_∞ contains j vertices, it has also the following interpretation.

Proposition 1. Let $i\in\mathbb{N}$ *be an arbitrary vertex of* $G_\infty=(\mathbb{N},E_\infty)$, then

$$p_j=\lim_{i\to\infty}\Pr\{(i-r,i)\in E_\infty,\ r=1,2,\cdots,j-1;\ (i-j,i)\notin E_\infty\}, \quad j=1,2,\cdots.$$

Proof. Trivial from (9).

So, in the long run p_j also equals the probability that an arbitrary customer has service conflicts with exactly $j-1$ of its immediate predecessors.

Let

$$F_i^+ = \{j \in \mathbb{N} \mid j \in \mathrm{Adj}\,(i),\ j > i\}, \qquad F_i^- = \{j \in \mathbb{N} \mid j \in \mathrm{Adj}\,(i),\ j < i\}$$

for $i \in \mathbb{N}$. Observe that F_i^+ consists of all customers arriving during $[T_i, T_i + Y_i)$ and F_i^- consists of all customers j arriving prior to customer i such that $T_j + Y_j > T_i$.

Proposition 2

(a) $$q_j \stackrel{\mathrm{d}}{=} \Pr\{|F_i^+| = j\} = E\{B(S_{j+1})\} - E\{B(S_j)\}, \qquad j = 0, 1, \cdots;$$

(b) $$\eta \stackrel{\mathrm{d}}{=} E\{|F_i^+|\} = \sum_{j=1}^{\infty} E[1 - B(S_j)] = E\{M(Y_i)\} < \infty,$$

in which $M(\cdot) = \sum_{j=1}^{\infty} A^{j*}(\cdot)$ is the renewal function pertaining to $\{X_i,\ i = 1, 2, \cdots\}$;

(c) $$\lim_{i \to \infty} E\{|F_i^-|\} = \eta = E\{|F_i^+|\}.$$

Proof. Since F_i^+ corresponds to all intervals with left endpoints in the interval $[T_i, T_i + Y_i)$, it follows that

$$q_j = \int_0^{\infty} \Pr\{S_j < t \leqq S_{j+1}\}\, dB(t) = E\{B(S_{j+1})\} - E\{B(S_j)\}.$$

Part (b) follows from

$$\eta = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} q_j = \sum_{j=1}^{\infty} E[1 - B(S_j)]$$

since $S_i \to \infty$ (a.s), as $i \to \infty$ and the last expression can be written as

$$\eta = \sum_{j=1}^{\infty} \int_0^{\infty} A^{j*}(t)\, dB(t) = \int_0^{\infty} M(t)\, dB(t) = E\{M(Y_i)\},$$

in which $A^{j*}(\cdot)$ denotes the j-fold convolution.

The finiteness of η is an easy consequence of the finiteness of EX_i and EY_i.

To prove (c), notice that

$$E\{|F_i^-|\} = \sum_{j=1}^{i-1} E[1 - B(S_{i-j})] = \sum_{j=1}^{i-1} E[1 - B(S_j)];$$

the assertion therefore follows from part (b) by letting $i \to \infty$.

We briefly discuss another scheme (referred to as the on-line scheme) to generate a stable, but not necessarily maximum, set in G_∞. This scheme $\tau = (\tau(1), \tau(2), \cdots)$, which actually characterises the customers served in the classical $GI/G/1/0$ system, is recursively defined as follows:

$$\tau(1)=1, \qquad \tau(i)=\min\{j\in\mathbb{N}: j\notin F^{+}_{\tau(i-1)}\}, \qquad i\geqq 2.$$

It is not difficult to see that $\{\tau(i),\ i=1,2,\cdots\}$ is a stable set. Let $C_i=\{\tau(i)\}\cup F^{+}_{\tau(i)}$, $i\geqq 1$, then the random variables $\{|C_i|,\ i=1,2,\cdots\}$ form a renewal sequence with lifetime distribution $\{q_{j-1},\ j=1,2,\cdots\}$. If Z_n^* denotes the cardinality of the generated stable set in G_n, then

$$\lim_{n\to\infty}\frac{1}{n}EZ_n^*=1/(1+\eta), \qquad \eta=\int_0^\infty M(t)\,dB(t), \tag{10}$$

giving the fraction of served customers in the long run for the classical $GI/G/1/0$ system.

It should be noticed that

$$\theta=\sum_{j=1}^{\infty}E\prod_{i=1}^{j}[1-B(S_i)]\leqq\sum_{j=1}^{\infty}E[1-B(S_j)]=\eta.$$

Observe that the perfect and the on-line schemes σ and τ coincide (a.s.) if $Y_i(\omega)=\mu^{-1}$, $i=1,2,\cdots$, therefore, in this case Z_n^* is maximum too, and $\theta=\eta$, exhibiting the fact that our optimisation problem makes no sense if the service times do not vary.

2. The system $GI/M/1/0$

In this section we consider the system $GI/M/1/0$ more closely. Let

$$B(t)=1-e^{-\mu t}, \qquad t\geqq 0, \qquad 0<\mu<\infty.$$

Introducing the Laplace–Stieltjes transform of $A(\cdot)$,

$$\alpha(s)=\int_0^\infty e^{-st}\,dA(t), \qquad s\geqq 0, \tag{11}$$

we obtain from (9) and (11)

$$p_j=[1-\alpha(j\mu)]\prod_{i=1}^{j-1}\alpha(i\mu), \qquad j=1,2,\cdots \tag{12}$$

and

$$\theta=\sum_{j=1}^{\infty}\prod_{i=1}^{j}\alpha(i\mu) \tag{13}$$

and thus

$$\lim_{n\to\infty}\frac{1}{n}EZ_n=1\Big/\Big\{1+\sum_{j=1}^{\infty}\prod_{i=1}^{j}\alpha(i\mu)\Big\}. \tag{14}$$

On the other hand, since (cf. Feller [2], p. 442),

$$\eta=\int_0^\infty M(t)\,dB(t)=\int_0^\infty e^{-\mu t}\,dM(t)=\frac{\alpha(\mu)}{1-\alpha(\mu)}, \qquad \mu>0$$

and we obtain

$$\lim_{n\to\infty}\frac{1}{n}EZ_n^*=1-\alpha(\mu). \tag{15}$$

Remark 1. Actually it is straightforward to derived from (3) and (4) for exponentially distributed interval lengths

$$\begin{aligned}\Pr\{v(1)=k,\ D_n=j\}&=\frac{1}{j}[1-\alpha(j\mu)]\prod_{i=1}^{j-1}\alpha(i\mu), \qquad k=1,2,\cdots,j,\\ &=\frac{1}{n}\prod_{i=1}^{n-1}\alpha(i\mu), \qquad k=1,2,\cdots,n,\end{aligned}$$

which also implies

$$\Pr\{v(1)=k \mid D_n=j\}=\frac{1}{j}, \qquad k=1,2,\cdots,j.$$

For the on-line scheme it can be shown that

$$\Pr\{Z_n^*=1+i\}=\binom{n-1}{i}[1-\alpha(\mu)]^i\alpha(\mu)^{n-i-1}, \qquad i=0,1,\cdots,n-1,$$

using the fact that (cf. Proposition 2(a))

$$q_j=\alpha(\mu)^j-\alpha(\mu)^{j+1}=[1-\alpha(\mu)]\alpha(\mu)^j.$$

The gain in using perfect information over the normal operation is best exposed by considering the systems under heavy traffic conditions. Let the interarrival intensity λ be fixed, while μ tends to zero so that the traffic offered $\rho=\lambda/\mu$ tends to infinity. First observe that, cf. (15),

$$\lim_{n\to\infty}\frac{1}{n}EZ_n^*\sim\frac{1}{\rho}, \qquad \rho\to\infty, \tag{16}$$

since $1-\alpha(\mu)\sim\mu EX_i$ as $\mu\downarrow 0$.

To determine the asymptotic behaviour EZ_n/n as $n\to\infty$, we first consider the system $D/M/1/0$. Since in this case $\alpha(s)=e^{-s/\lambda}$, (13) yields

$$\theta=\sum_{j=1}^{\infty}\prod_{i=1}^{j}e^{-i/\rho}=\sum_{j=1}^{\infty}\exp\left\{-\frac{1}{2\rho}j(j+1)\right\}, \qquad \rho\overset{\mathrm{d}}{=}\frac{\lambda}{\mu}. \tag{17}$$

Lemma 1

$$h(s)=\sum_{j=0}^{\infty}\exp\left\{-\frac{s}{2}j(j+1)\right\}\sim\sqrt{\frac{\pi}{2s}}, \qquad s\downarrow 0. \tag{18}$$

Proof. Observe that

$$h(s)=\sum_{j=0}^{\infty}\exp\left\{-\frac{s}{2}j(j+1)\right\}=\tfrac{1}{2}\sum_{j=-\infty}^{\infty}\exp\left\{-\frac{s}{2}j(j+1)\right\}, \qquad s>0.$$

By Jacobi's imaginary transformation for theta-functions (see [7], p. 476), the series on the right equals

$$h(s)=\tfrac{1}{2}\cdot\sqrt{\frac{2\pi}{s}}\cdot\sum_{j=-\infty}^{\infty}\exp\left\{-\frac{2}{s}\left(j\pi+\frac{s}{4i}\right)^2\right\},\qquad s>0.$$

Hence,

$$h(s)=\sqrt{\frac{\pi}{2s}}\cdot e^{s/8}\left\{1+2\sum_{j=1}^{\infty}\exp\left(-\frac{2\pi^2}{s}j^2\right)\cos j\pi\right\}$$

and the assertion follows by letting $s\downarrow 0$.

Since $1+\theta=h(\rho^{-1})$, the heavy traffic result for the system $D/M/1/0$ follows from (14) and (18), with $s=\rho^{-1}$.

Corollary. For the system $D/M/1/0$

$$\lim_{n\to\infty}\frac{1}{n}EZ_n\sim\sqrt{\frac{2}{\pi\rho}},\qquad \rho=\frac{\lambda}{\mu}\to\infty. \tag{19}$$

As it turns out relation (19) holds more generally. In order to show that it also holds for the system $GI/M/1/0$ we need two preparatory lemmas.

Lemma 2. *Let $f(s)=\alpha(s)-e^{-s/\lambda}$, $s\geqq 0$, where $EX_i=1/\lambda$. The function f satisfies*

(a)
$$f(s)\geqq 0,\qquad s\geqq 0$$

(b)
$$0\leqq\max_{s\geqq 0}f(s)\leqq\delta=A(1/\lambda-0)<1.$$

Proof. By Jensen's inequality (see Feller [2], p. 151),

$$\alpha(s)=E\exp(-sX_i)\geqq\exp(-s/\lambda),\qquad s\geqq 0$$

implying that $f(s)\geqq 0$.

Observe that $f(0)=0$ and $f(s)\downarrow 0$ if $s\to\infty$. It is well known that $\alpha(s)$ and $e^{-s/\lambda}$ are completely monotone decreasing functions both strictly less than 1 for $s>0$. Hence f must attain a global maximum strictly less than 1. In fact we have

$$f(s)=\int_{[0,1/\lambda)}(e^{-st}-e^{-s/\lambda})\,dA(t)+\int_{[1/\lambda,\infty)}(e^{-st}-e^{-s/\lambda})\,dA(t)\leqq A\left(\frac{1}{\lambda}-0\right)<1.$$

Lemma 3. *There exists a finite non-negative constant K such that*

$$h(s)\leqq\theta=1+\sum_{j=1}^{\infty}\prod_{i=1}^{j}\alpha(is)\leqq h(s)+K,\qquad s>0.$$

Proof. Assuming $EX_i=1$, Jensen's inequality gives

$$1+\sum_{j=1}^{\infty}\prod_{i=1}^{j}\alpha(is)=1+\sum_{j=1}^{\infty}E\left\{\exp\left(-s\sum_{1}^{j}iX_i\right)\right\}\geqq\sum_{j=0}^{\infty}\exp\left\{-\frac{s}{2}j(j+1)\right\}$$

which proves the first inequality.

Next consider the difference

$$1+\sum_{j=1}^{\infty}\prod_{i=1}^{j}\alpha(is)-\sum_{j=0}^{\infty}\exp\left\{-\frac{s}{2}j(j+1)\right\}=\sum_{j=1}^{\infty}\prod_{i=1}^{j}[\alpha(is)-e^{-is}]$$
$$=\sum_{j=1}^{\infty}\prod_{i=1}^{j}f(is)$$
$$\leqq\sum_{j=1}^{\infty}\prod_{i=1}^{j}\delta=\frac{\delta}{1-\delta}=K$$

by virtue of Lemma 2, and the proof is complete.

Now from the Lemmas 1 and 3 we immediately have the following result, recalling Theorem 2 and (13), and taking $s=1/\rho$.

Theorem 3. *Let $\rho=\lambda/\mu$, then for the system $GI/M/1/0$*

$$\lim_{n\to\infty}\frac{1}{n}EZ_n=\frac{1}{1+\theta}\sim\sqrt{\frac{2}{\pi\rho}},\qquad \rho\to\infty.$$

It should be observed from Lemma 3 that $h(\rho^{-1})$ provides a lower bound for $1+\theta$, expressing the fact that the system $D/M/1/0$ is the best among all $GI/M/1/0$ systems in achieving a high throughput rate. In the case of the usual (on-line) operation of a $GI/M/1/0$ system under heavy traffic conditions, the fraction of served customers behaves as ρ^{-1}, $\rho\to\infty$, see (15), which is to be contrasted with $O(1/\sqrt{\rho})$ attainable with perfect information.

Table 1 illustrates for the systems $M/M/1/0$ and $D/M/1/0$ the behaviour of the fraction of served customers as a function of the offered traffic ρ under the usual (on-line) operation and under perfect information. The gain from perfect informa-

TABLE 1
The fraction of served customers

	$M/M/1$		$D/M/1$		
ρ	on-line	perf.	on-line	perf.	$(2/(\pi\rho))^{1/2}$
0·2	0·833	0·845	0·993	0·993	
0·5	0·667	0·709	0·865	0·879	
0·8	0·556	0·624	0·713	0·763	
1·0	0·500	0·582	0·632	0·704	0·798
2	0·333	0·456	0·393	0·530	0·564
3	0·250	0·388	0·281	0·442	0·461
4	0·200	0·345	0·221	0·387	0·399
5	0·167	0·314	0·181	0·348	0·357
8	0·111	0·255	0·118	0·278	0·282
10	0·091	0·231	0·095	0·249	0·252
20	0·048	0·168	0·049	0·177	0·178
50	0·020	0·109	0·020	0·113	0·113

tion is small for light traffic, i.e. small ρ, since there are almost no service conflicts between customers. In fact, when $\mu \to \infty$, it is readily verified from (14) that $1/(1+\theta) \sim 1-\alpha(\mu) = 1/(1+\eta)$, cf. (15).

Remark 2. Although it is outside the scope of the present paper, it is interesting to point out a result obtainable from the above lemmas. In [3] the asymptotic behaviour of $\tilde{M}(t) = \sum_{j=1}^{\infty} \Pr\{X_1 + 2X_2 + \cdots + jX_j \leqq t\}$, $t \to \infty$, has been obtained in the exponential case by several authors, showing that $\tilde{M}(t) \sim \sqrt{2\lambda t}$. Stam (personal communication) actually shows that this holds more generally, provided $EX_i^4 < \infty$. Observing that θ is essentially the Laplace–Stieltjes transform of $\tilde{M}(t)$ and noting that $\tilde{M}(t)$ is non-decreasing, it follows from the Lemmas 2 and 3 and a Tauberian theorem for Laplace–Stieltjes transforms (cf. Widder [8], p. 192], that $\tilde{M}(t) \sim \sqrt{2\lambda t}$, $t \to \infty$, under the sole assumption $EX_i < \infty$.

References

[1] FELLER, W. (1957) *An Introduction to Probability Theory and its Applications,* Vol. I, 2nd edn. Wiley, New York.

[2] FELLER, W. (1966) *An Introduction to Probability Theory and its Applications,* Vol. II. Wiley, New York.

[3] FRENK, H. AND STEUTEL, F. W. (1988) Problem 211, *Statist. Neerlandica* **42,** 145.

[4] GOLUMBIC, M. C. (1980) *Algorithmic Graph Theory and Perfect Graphs.* Academic Press, New York.

[5] NAWIJN, W. M. (1990) Look-ahead policies for admission to a single server loss system. *Operat. Res.* **5,** 854–862.

[6] ROSS, S. M. (1970) *Applied Probability Models with Optimization Applications.* Holden-Day, San Francisco.

[7] WHITTAKER, E. T. AND WATSON, G. N. (1962) *Modern Analysis,* 4th edn. Cambridge University Press.

[8] WIDDER, D. V. (1946) *The Laplace Transform.* Princeton University Press.